

VERY STRONG APPROXIMATION FOR CERTAIN ALGEBRAIC VARIETIES

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ABSTRACT. Let F be a global field. In this work, we show that the Brauer-Manin condition on adelic points for subvarieties of a torus T_F over F cuts out exactly the rational points, if either F is a function field or, if $F = \mathbb{Q}$ and T_F is split. As an application, we prove a conjecture of Harari-Voloch over global function fields which states, roughly speaking, that on any rational hyperbolic curve, the local integral points with the Brauer-Manin condition are the global integral points. Finally we prove for tori over number fields a theorem of Stoll on adelic points of zero-dimensional subvarieties in abelian varieties.

1. INTRODUCTION

Since the rational points of an algebraic variety over a global field satisfy the Brauer-Manin conditions, it is natural to ask whether, conversely, these conditions determine the rational points. There is some recent progress for projective curves and abelian varieties ([19], see also [23]). On the other hands, Colliot-Thélène and the second named author introduced in [3] the study of integral points for homogeneous spaces of linear algebraic groups in relation with strong approximation properties. Afterward, Harari and Voloch [12] considered the integral points on hyperbolic curves satisfying some Brauer-Manin conditions. In particular they conjectured that these conditions determine the integral points of rational hyperbolic curves. In the present paper, the main purpose is first to pursue the study of integral points and strong approximation properties for general algebraic varieties with focus on subvarieties of torus. Secondly, we solve Harari-Voloch conjecture for global function fields.

Now let us fix some notation and give some definition. Let F be a global field, Ω_F be the set of all primes in F , ∞_F be the set of all Archimedean primes of F ($\infty_F = \emptyset$ if F is a function field). We will denote by \mathfrak{o}_S the ring of S -integers for a *non-empty* finite subset S of Ω_F containing ∞_F . If F is a number field, we denote by \mathfrak{o}_F the ring of integers of F . For each finite prime $v \in \Omega_F$, $\mathfrak{o}_{F,v}$ denotes the discrete valuation ring of F associated to v , \mathfrak{o}_v its completion, $F_v = \text{Frac}(\mathfrak{o}_v)$. If $v \in \infty_F$, we put $\mathfrak{o}_v = \mathfrak{o}_{F,v} = F_v$.

Let \mathbb{A}_F be the adelic ring of F . We denote by \mathbb{A}_F^S the S -adeles obtained by projecting \mathbb{A}_F to $\prod_{v \notin S} F_v$ endowed with the induced adelic topology and have

$$\mathbb{A}_F = \left(\prod_{s \in S} F_v \right) \times \mathbb{A}_F^S$$

for any finite subset S of Ω_F containing ∞_F .

An *algebraic variety over F* will be a separated scheme of finite type over F . Let X_F be an algebraic variety. For all $v \in \Omega_F$, $X_F(F_v)$ is endowed with the topology

Date: September 19, 2014.

2010 Mathematics Subject Classification. 14G05, 14G25, 11G35, 14F22.

Key words and phrases. Strong approximation, Brauer-Manin obstruction, integral model.

induced by the topology of F_v . The set of adelic points

$$X_F(\mathbb{A}_F) := \text{Mor}_F(\text{Spec } \mathbb{A}_F, X_F) \subset \prod_{v \in \Omega_F} X_F(F_v)$$

can be canonically endowed with a locally compact Hausdorff topology ([5], §4). When X_F is affine, X is a closed subvariety of an affine space. Then the adelic topology of $X_F(\mathbb{A}_F)$ is induced by the product of usual adelic topology under the closed immersion.

Let

$$\text{Br}(X_F) = H_{\text{ét}}^2(X_F, \mathbb{G}_m) \quad \text{and} \quad \xi \in \text{Br}(X_F).$$

For any $(x_v)_{v \in \Omega_F} \in X_F(\mathbb{A}_F)$, one has $\text{inv}_v(\xi(x_v)) = 0$ for almost all $v \in \Omega_F$ (see [21] §5.2), where

$$\text{inv}_v : \text{Br}(F_v) \rightarrow \mathbb{Q}/\mathbb{Z}$$

is the local invariant map. For any subgroup H of $\text{Br}(X_F)$, define

$$X_F(\mathbb{A}_F)^H = \{(x_v)_v \in X_F(\mathbb{A}_F) : \sum_{v \in \Omega_F} \text{inv}_v(\xi(x_v)) = 0 \text{ for all } \xi \in H\}.$$

The Artin reciprocity law implies that

$$X_F(F) \subseteq X_F(\mathbb{A}_F)^{\text{Br}(X_F)} \subseteq X_F(\mathbb{A}_F).$$

For each ξ , $\text{inv}_v(\xi(-))$ is locally constant, hence continuous, on $X_F(F_v)$. This implies that $X_F(\mathbb{A}_F)^H$ is a closed subset of $X_F(\mathbb{A}_F)$.

Part of the following definition is taken from [4], Definition 2.1.

Definition 1.1 Let S be a finite subset of Ω_F and let

$$P_S : X_F(\mathbb{A}_F) \longrightarrow X_F(\mathbb{A}_F^S)$$

be the projection map. We will say that *the strong approximation property with Brauer-Manin obstruction off S holds for X_F* or that *X_F satisfies the strong approximation property with Brauer-Manin obstruction off S* (SAP-BM off S for short) if $X_F(F)$, under the diagonal map, is dense in $P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}]$ with adelic topology.

We will say that *the very strong approximation property with Brauer-Manin obstruction off S holds for X_F* or that *X_F satisfies the very strong approximation property with Brauer-Manin obstruction off S* (abbreviated to VSAP-BM off S) if

$$X_F(F) = P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}].$$

In the classical strong approximation examples (*e.g.*, affine spaces and simply connected semi-simple linear algebraic groups), the Brauer groups are always constant (*i.e.*, the canonical map $\text{Br}(F) \rightarrow \text{Br}(X_F)$ is surjective) and $X_F(\mathbb{A}_F)^{\text{Br}(X_F)} = X_F(\mathbb{A}_F)$. So the above definition extends the classical situations.

The first main result of this work deals with VSAP-BM and can be regarded as an analogue of Theorem D in [19] for tori.

Theorem 1.2. (Corollary 3.6) *Let X_F be a subvariety of a torus T_F over a global field F . Then X_F satisfies VSAP-BM off S for any finite subset S of Ω_F containing ∞_F if one of the following conditions is satisfied:*

- (1) F is a global function field (then $\infty_F = \emptyset$);
- (2) $F = \mathbb{Q}$ or an imaginary quadratic field and T_F is split over F .

For general algebraic varieties, we have the following result.

Theorem 1.3. (Corollary 3.10) *Any algebraic variety over \mathbb{Q} or an imaginary quadratic field always contains a dense open subvariety which satisfies VSAP-BM off S for any finite subset S of Ω_F containing ∞_F .*

As an application, we prove Harari-Voloch's conjecture over a global function field by using [24], Theorem 1.

Theorem 1.4. (Corollary 4.7) *Let F be a global function field and X_F be an open subset of \mathbf{P}_F^1 with complementary of degree ≥ 3 as a reduced separable divisor. Then*

$$X_F(F) = X_F(\mathbb{A}_F^S)^{B_S(X_F)},$$

where

$$B_S(X_F) = \text{Ker}[\text{Br}(X_F) \rightarrow \prod_{v \in S} \text{Br}(X_{F_v}) / \text{Br}(F_v)]$$

for any finite subset S of Ω_F .

We prove that the same result is also true over \mathbb{Q} when $S = \infty_{\mathbb{Q}}$ and $\mathbf{P}_{\mathbb{Q}}^1 \setminus X_{\mathbb{Q}}$ contains more than a single point in Corollary 5.11.

The next theorem can be regarded as an analogue for tori of [23], Theorem 3.11. After this paper was written, we learned that C.-L. Sun proved an equivalent statement ([25], Theorem 1) by a slightly different method.

Theorem 1.5. (Theorem 5.7) *If Z is a finite subscheme of a torus T_F over a number field F , then for any finite subset S of Ω_F containing ∞_F , we have*

$$\overline{T_F(F)}^S \cap Z(\mathbb{A}_F^S) = Z(F)$$

where $\overline{T_F(F)}^S$ is the topological closure of $T_F(F)$ inside $T_F(\mathbb{A}_F^S)$.

Note that if this result is true for any curve contained in T_F , then Harari-Voloch conjecture holds over number fields (Proposition 4.6 (1)). As an application of the above theorem, we prove in Proposition 5.9 that if $f : X_F \rightarrow Y_F$ is a quasi-finite morphism of rational hyperbolic curves and if Harari-Voloch conjecture is true for Y_F , then it is true for X_F .

2. INTEGRAL MODELS AND STRONG APPROXIMATION

Let F, S and X_F be as in § 1. The aim of this section is first to show that the subsets of $X_F(\mathbb{A}_F^S)$ of the form $\prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v)$, when \mathbf{X} runs through the integral models of X_F over \mathfrak{o}_S , form a topology basis for $X_F(\mathbb{A}_F^S)$ (Corollary 2.9). Then we apply this result to relate strong approximation of adelic points to that of integral points (Theorem 2.10).

2.1. Integral models. We start with some basic definitions and technical preliminary results.

Definition 2.1 Let S be a non-empty finite subset of Ω_F containing ∞_F . An *integral model* (or a *model*) of X_F over \mathfrak{o}_S is a scheme \mathbf{X} faithfully flat of finite type and separated over \mathfrak{o}_S endowed with an isomorphism $\mathbf{X} \times_{\mathfrak{o}_S} F \cong X_F$ over F . An *integral point* on \mathbf{X} is a section $\in \mathbf{X}(\mathfrak{o}_S)$.

Define $\mathbb{A}_{F,S} = \prod_{v \in S} F_v \times \prod_{v \notin S} \mathfrak{o}_v \subset \mathbb{A}_F$. Then $\mathbf{X}(\mathbb{A}_{F,S}) \subseteq X_F(\mathbb{A}_F)$ and it is known that the canonical map

$$\mathbf{X}(\mathbb{A}_{F,S}) \rightarrow \prod_{v \in S} \mathbf{X}(F_v) \times \prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v) = \prod_{v \in S} X_F(F_v) \times \prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v)$$

is a bijection ([5], Theorem 3.6, where \mathcal{O}_v is our \mathfrak{o}_v). By Artin reciprocity law, a necessary condition for $\mathbf{X}(\mathfrak{o}_S) \neq \emptyset$ is

$$\mathbf{X}(\mathbb{A}_{F,S})^{\mathrm{Br}(X_F)} \neq \emptyset \quad (2.1)$$

When $\mathrm{Br}(X_F)$ is constant, the above condition is equivalent to the classical local-global (Hasse) principle.

Definition 2.2 If the condition (2.1) is also sufficient to insure that $\mathbf{X}(\mathfrak{o}_S) \neq \emptyset$, we say the *Brauer-Manin obstruction is the only obstruction* for the existence of integral points on \mathbf{X} .

We will use the following gluing process several times.

Lemma 2.3. *Let $v_1, \dots, v_n \in \mathrm{Spec}(\mathfrak{o}_S)$ and $V = \mathrm{Spec} \mathfrak{o}_S \setminus \{v_1, \dots, v_n\}$. Suppose we are given separated schemes of finite type \mathbf{Y} over V , \mathbf{Y}_i over \mathfrak{o}_{F,v_i} and isomorphisms $f_i : \mathbf{Y}_F \simeq (\mathbf{Y}_i)_F$. Then there exists a unique (up to isomorphisms) separated scheme of finite type \mathbf{X} over \mathfrak{o}_S such that $\mathbf{X}_V \simeq \mathbf{Y}$ and $\mathbf{X}_{\mathfrak{o}_{F,v_i}} \simeq \mathbf{Y}_i$.*

If \mathbf{Y} , $\mathbf{Y}_1, \dots, \mathbf{Y}_n$ are group schemes and if the f_i are isomorphisms of group schemes, then \mathbf{X} is a group scheme over \mathfrak{o}_S .

Proof. Let $f_{ij} = f_j \circ f_i^{-1} : (\mathbf{Y}_i)_F \rightarrow (\mathbf{Y}_j)_F$. Then $f_{jk} \circ f_{ij} = f_{ik}$. There are open neighborhoods $V_i \ni v_i$ such that \mathbf{Y}_i and f_i extend to V_i and the identity $f_{jk} \circ f_{ij} = f_{ik}$ holds on $V_i \cap V_j \cap V_k$. We can then use the usual gluing process to construct \mathbf{X} . The separatedness of \mathbf{X} comes from that of $\mathbf{Y} \rightarrow V$ and of $\mathbf{Y}_i \rightarrow \mathrm{Spec} \mathfrak{o}_{F,v_i}$. The assertion on the structure of group scheme is straightforward. \square

Remark 2.4 Let \mathbf{Y} be an \mathfrak{o}_S -scheme of finite type, not necessarily flat, with generic fiber isomorphic to X . Let \mathbf{X} be the biggest flat closed subscheme of \mathbf{Y} (defined by the sheaf of ideals of \mathfrak{o}_S -torsion elements of $\mathcal{O}_{\mathbf{X}}$ or, equivalently, \mathbf{X} is the scheme-theoretic closure in \mathbf{Y} of the generic fiber of \mathbf{Y}). Then for any flat \mathfrak{o}_S -scheme T , the canonical map $\mathbf{Y}(T) \rightarrow \mathbf{X}(T)$ is bijective. So for the questions we deal with in this paper, it is harmless to restrict ourselves to flat models. On the other hand, for a model \mathbf{X} to have local sections in \mathfrak{o}_v for all $v \notin S$, it is necessary that \mathbf{X} be faithfully flat over \mathfrak{o}_S .

Proposition 2.5. *Let R be a Dedekind domain with field of fractions K . Let X_K be an algebraic variety over K . Then X_K has a model \mathbf{X} over R . Moreover there exists a such \mathbf{X} which is proper (resp. projective) over R if X_K is proper (resp. projective) over K .*

Proof. The case when X_K is projective is immediate: embed X_K into a projective space \mathbb{P}_K^N and take the scheme-theoretic closure \mathbf{X} of X_K in \mathbb{P}_R^N . As \mathbf{X} is proper and flat over R , it is faithfully flat.

Now suppose X_K is affine. Note that if we extend directly to R by scaling a system of equations defining X_K and take the biggest closed subscheme flat over R , we just get a flat, but not necessarily faithfully flat scheme over R . Let $Y_K \subseteq \mathbb{P}_K^N$ be a projective variety containing X_K as a dense open subvariety. Consider the scheme-theoretic closure \mathbf{Y} of Y_K in \mathbb{P}_R^N and \mathbf{Z} the scheme-theoretic closure of $Y_K \setminus X_K$ in \mathbf{Y} . Then $\mathbf{X} := \mathbf{Y} \setminus \mathbf{Z}$ is flat, separated and finite type over R , with generic fiber equal to X_K . Let us show \mathbf{X} is faithfully flat. As \mathbf{Y}, \mathbf{Z} are flat and proper over R , they are faithfully flat and

$$\dim \mathbf{Z}_v = \dim \mathbf{Z}_K < \dim \mathbf{Y}_K = \dim \mathbf{Y}_v, \quad \forall v \in \mathrm{Spec} R.$$

Therefore $\mathbf{X}_v = \mathbf{Y}_v \setminus \mathbf{Z}_v \neq \emptyset$ and \mathbf{X} is faithfully flat.

In the general case, as X_K is separated and of finite type over K , it extends to a separated scheme \mathbf{X}_1 of finite type and faithful flat over some open subset $V \subseteq \operatorname{Spec} R$ ([8], IV.8.8.2(ii)). Let U_K be a non-empty affine open subset of X_K . By the affine case, there exists a faithfully flat model \mathbf{U} of U_K over R . Shrinking V if necessary, we can suppose $U_K \subseteq X_K$ extends to an open immersion $\mathbf{U}_V \hookrightarrow (\mathbf{X}_1)_V$. Let \mathbf{X} be the gluing of \mathbf{X}_1 and \mathbf{U} along \mathbf{U}_V . Then \mathbf{X} is a model of X_K over R .

Suppose X_K is proper. The model \mathbf{X} we obtained above is not proper in general. But by Nagata's embedding theorem ([14], [6]), \mathbf{X} is an open subscheme of some proper scheme \mathbf{Z} over R . As \mathbf{X} is flat, it is contained in the scheme-theoretic closure \mathbf{Z}' of X_F in \mathbf{Z} . Now \mathbf{Z}' is a proper model of \mathbf{X} (it is faithfully flat over R by properness). \square

Remark 2.6 Suppose X_F is smooth. To study the integral points of X_F over \mathfrak{o}_S when X_F has local points at every $v \notin S$, we can restrict ourselves to smooth models in the sense that for any integral model \mathbf{X} such that $\mathbf{X}(\mathfrak{o}_v) \neq \emptyset$ for all $v \notin S$, there exists a quasi-projective morphism $\tilde{\mathbf{X}} \rightarrow \mathbf{X}$ of models of X_F such that $\tilde{\mathbf{X}}$ is smooth and the canonical maps $\tilde{\mathbf{X}}(\mathfrak{o}_S) \rightarrow \mathbf{X}(\mathfrak{o}_S)$ and $\tilde{\mathbf{X}}(\mathfrak{o}_v) \rightarrow \mathbf{X}(\mathfrak{o}_v)$, $v \notin S$, are bijective. Indeed, \mathbf{X} is already smooth over some dense open subset V of $\operatorname{Spec}(\mathfrak{o}_S)$. By the gluing process 2.3, we are reduced to the local case. Then the result follows from the smoothening procedure, see [1], Theorems 3.1/3: if $\mathbf{X}' \rightarrow \mathbf{X}$ is a smoothening, then take $\tilde{\mathbf{X}}$ equal to the smooth locus of \mathbf{X}' . Note that for any closed point $v \in \operatorname{Spec}(\mathfrak{o}_S)$, we have $\tilde{\mathbf{X}}(\mathfrak{o}_v) = \mathbf{X}(\mathfrak{o}_v)$ because the smoothening commutes with base changes of index 1 (see [1], Corollary 3.6/6). In particular $\tilde{\mathbf{X}}(\mathfrak{o}_v) \neq \emptyset$ and $\tilde{\mathbf{X}}$ is faithfully flat over \mathfrak{o}_S . As a counterpart, if we start with a proper model \mathbf{X} , the smoothening will produce a non-proper smooth model in general.

If X_F is furthermore a smooth algebraic group and the model \mathbf{X} is a (separated) group scheme, then $\tilde{\mathbf{X}}$ can be chosen to be a smooth group scheme. This is a consequence of the existence of group smoothening as shown in [1], 7.1/5. The existence over a global Dedekind base scheme follows from the local case by the gluing process as above. We also have $\tilde{\mathbf{X}}(\mathfrak{o}_v) = \mathbf{X}(\mathfrak{o}_v)$ for all closed points v because the group smoothening is obtained by successive dilatations and the latter operations commute with the passage to the completion.

Lemma 2.7. *Let \mathbf{X} be an integral model of X_F over \mathfrak{o}_S . Then the canonical commutative diagram of injective maps*

$$\begin{array}{ccc} \mathbf{X}(\mathfrak{o}_S) & \longrightarrow & \mathbf{X}(\mathbb{A}_{F,S}) \\ \downarrow & & \downarrow \\ X_F(F) & \longrightarrow & \prod_{v \in \Omega_F} X_F(F_v) \end{array}$$

allows us to identify $\mathbf{X}(\mathfrak{o}_S)$ with $X_F(F) \cap \mathbf{X}(\mathbb{A}_{F,S})$.

Proof. Let $\sigma \in X_F(F)$ such that $\sigma \in \mathbf{X}(\mathfrak{o}_v)$ in $X_F(F_v)$ for all $v \notin S$. Then $\sigma \in \mathbf{X}(\mathfrak{o}_{F,v})$ for all $v \notin S$. Moreover σ extends to a $U(v) \rightarrow \mathbf{X}$ for some open neighborhood $U(v)$ of $v \in \operatorname{Spec} \mathfrak{o}_S$ for all $v \notin S$. One concludes that $\sigma \in \mathbf{X}(\mathfrak{o}_S)$ by gluing these morphisms. \square

Let B be an integral domain containing \mathfrak{o}_S , with field of fractions L . Let \mathbf{X} be an integral model of X_F over \mathfrak{o}_S . The base change $\operatorname{Spec} B \rightarrow \operatorname{Spec} L$ induces canonically a map $\mathbf{X}(B) \rightarrow X_F(L)$ (depending on the isomorphism $\mathbf{X}_F \simeq X_F$) which is injective

because \mathbf{X} is separated over \mathfrak{o}_S . For simplicity, we will consider $\mathbf{X}(B)$ as a subset of $X_F(L)$. By Lemma 2.7, $\mathbf{X}(\mathfrak{o}_S)$ can be identified to $X_F(F) \cap \mathbf{X}(\mathbb{A}_{F,S})$.

Next we relate the analytic open subsets to integral points of models.

Proposition 2.8. *Let R be a discrete valuation ring with field of fractions K . Let \hat{R} be its completion and $\hat{K} = \text{Frac}(\hat{R})$. Let X_K be an algebraic variety over K , let $q \in X_K(\hat{K})$ and let $U \subseteq X_K(\hat{K})$ be an open neighborhood of q for the topology induced by that of \hat{K} . Then there exists an integral model \mathbf{X} of X_K over R such that $q \in \mathbf{X}(\hat{R}) \subseteq U$.*

Proof. First let us prove that if \mathbf{X}_0 is a model of X_K over R such that $q \in \mathbf{X}_0(\hat{R})$, then there exists a finite sequence of dilatations ([1], §3.2)

$$\mathbf{X}_d \rightarrow \mathbf{X}_{d-1} \rightarrow \cdots \rightarrow \mathbf{X}_0$$

such that $\mathbf{X}_d(\hat{R}) \subseteq U$. Let $x_0 \in (\mathbf{X}_0)_k(k)$ be the specialization of q (i.e. the image of the closed point of $\text{Spec}(\hat{R})$ by $q : \text{Spec}(\hat{R}) \rightarrow \mathbf{X}_0$). Consider $\mathbf{X}_1 \rightarrow \mathbf{X}_0$ the dilatation of $\{x_0\} = \text{Spec } k(x_0)$ on \mathbf{X}_0 (see [1], §3.2). By the universal property of the dilatation, the image of the (injective) canonical map $\mathbf{X}_1(\hat{R}) \rightarrow \mathbf{X}_0(\hat{R})$ consists of the points of $X_K(\hat{K})$ whose specializations in $(\mathbf{X}_0)_k(k)$ are equal to x_0 . In particular, $q \in \mathbf{X}_1(\hat{R})$. We define inductively a sequence $\mathbf{X}_{i+1} \rightarrow \mathbf{X}_i$ of dilatation of x_i on \mathbf{X}_i , where $x_i \in (\mathbf{X}_i)_k(k)$ is the specialization of $q \in \mathbf{X}_i(\hat{R})$. Let us show that $\mathbf{X}_d(\hat{R}) \subseteq U$ when d is big enough. As the dilatation commutes with flat base change ([1], 3.2/2(b)), we can suppose R is complete.

By construction, for any closed subscheme Z_i of $(\mathbf{X}_i)_k$ and for any open subscheme \mathbf{W} of \mathbf{X}_i , the dilatation of $Z_i \cap \mathbf{W}$ on \mathbf{W} is the restriction to \mathbf{W} of the dilatation $\mathbf{X}_{i+1} \rightarrow \mathbf{X}_i$. So we can suppose \mathbf{X}_0 , and hence X_K , are affine. We have a closed immersion $\iota : \mathbf{X}_0 \hookrightarrow \mathbf{A}_0 := \text{Spec } R[x_1, \dots, x_n]$ such that $\iota(q)$ is the origin $(0, \dots, 0)$. Let t be a uniformizing element of R . For any integer $i \geq 0$, let $D(0, i) = t^i R^n \subset \mathbf{A}_0(R)$. Then by the definition of the topology on $X_K(K)$, there exists $d \geq 0$ such that $\iota_K^{-1}(D(0, d)) \subseteq U$.

By the construction of the dilatation ([1], 3.2/2(b)), ι induces a closed immersion

$$\iota_d : \mathbf{X}_d \hookrightarrow \mathbf{A}_d := \text{Spec } R[x_1/t^d, \dots, x_n/t^d]$$

and

$$\mathbf{X}_d(R) \subseteq \iota_d^{-1}(\mathbf{A}_d(R)) \cap X_K(K) = \iota_K^{-1}(D(0, d)) \cap X_K(K) \subseteq U.$$

Now to prove the proposition, it is enough to find a model \mathbf{X}_0 of X_K over R such that $q \in \mathbf{X}_0(\hat{R})$. Let $W_K \subseteq X_K$ be an affine Zariski open neighborhood of $q \in X_K(\hat{K})$. Then W_K extends to an affine model \mathbf{W} . Up to scaling a system of coordinates on W_K , we can suppose that $q \in \mathbf{W}(\hat{R})$. Now we can glue X_K and \mathbf{W} along W_K to get an integral model \mathbf{X}_0 of X_K over R with $q \in \mathbf{X}_0(\hat{R})$. \square

Corollary 2.9. *Let X_F be an algebraic variety over F . Let $U \subseteq X_F(\mathbb{A}_F^S)$ be an open subset and let $z \in U$. Then there exists an integral model \mathbf{X} of X_F over \mathfrak{o}_S such that*

$$z \in \prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v) \subseteq U.$$

In other words, the subsets $\prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v) \subset X_F(\mathbb{A}_F^S)$, where \mathbf{X} runs through the integral models of X_F over \mathfrak{o}_S , form a basis of the topological space $X_F(\mathbb{A}_F^S)$.

Moreover, if X_F is a (smooth) algebraic group over F , then the subsets of the form $\prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v)$, where \mathbf{X} is a (smooth) group scheme model of X_F over \mathfrak{o}_S , form a fundamental system of neighborhood of 1.

Proof. By the definition of the adelic topology ([5], §4), there exists an integral model \mathbf{Y} of X_F over \mathfrak{o}_T for some finite subset $T \supseteq S$ of Ω_F such that

$$U \supseteq \prod_{v \in T \setminus S} U_v \times \prod_{v \notin T} \mathbf{Y}(\mathfrak{o}_v)$$

for some non-empty open subsets U_v of $X_F(F_v)$.

Write $z = (z_v)_{v \notin S}$. For any $v \in T \setminus S$, there exists an integral model \mathbf{X}_v of X_F over $\mathfrak{o}_{F,v}$ such that $z_v \in \mathbf{X}_v(\mathfrak{o}_v) \subseteq U_v$ by Proposition 2.8(1). Using Lemma 2.3, we obtain an integral model \mathbf{X} of X_F over \mathfrak{o}_S as required by gluing the schemes \mathbf{X}_v , $v \in T \setminus S$ with $\mathbf{Y} \times_{\mathfrak{o}_S} \mathfrak{o}_T$.

The case of algebraic groups is proved in a similar way using dilatations along the unit section, see Remark 2.6. \square

2.2. Applications to strong approximations. Now we can give a Diophantine interpretation of strong approximation.

Theorem 2.10. *Let X_F be an algebraic variety over F and let S be a non-empty finite subset of Ω_F containing ∞_F . Then X_F satisfies SAP-BM off S if and only if the Brauer-Manin obstruction is the only obstruction for the existence of integral points for every integral model \mathbf{X} of X_F over \mathfrak{o}_S .*

Proof. (\Rightarrow) Let \mathbf{X} be an integral model of X_F over \mathfrak{o}_S and suppose that

$$\mathbf{X}(\mathbb{A}_{F,S})^{\text{Br}(X_F)} \neq \emptyset.$$

This set being open (with respect to the adelic topology) in $X_F(\mathbb{A}_F)^{\text{Br}(X_F)}$, we have

$$\mathbf{X}(\mathfrak{o}_S) = X_F(F) \cap P_S[\mathbf{X}(\mathbb{A}_{F,S})^{\text{Br}(X_F)}] \neq \emptyset$$

by hypothesis on X_F . Hence the Brauer-Manin obstruction is the only obstruction for the existence of integral points on \mathbf{X} .

(\Leftarrow) Let U be an open subset of $X_F(\mathbb{A}_F^S)$ such that

$$U \cap P_S(X_F(\mathbb{A}_F)^{\text{Br}(X_F)}) \neq \emptyset.$$

Let z be a point of this set. By Corollary 2.9, there exists an integral model \mathbf{X} over \mathfrak{o}_S such that

$$z \in \prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v) \subseteq U.$$

This implies that

$$\mathbf{X}(\mathbb{A}_{F,S})^{\text{Br}(X_F)} \neq \emptyset.$$

By the hypothesis, one obtains that $\mathbf{X}(\mathfrak{o}_S) \neq \emptyset$. As the natural map $\mathbf{X}(\mathfrak{o}_S) \rightarrow \mathbf{X}(\mathbb{A}_F^S)$ has image in $\prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v) \subseteq U$ and $\mathbf{X}(\mathfrak{o}_S) \subseteq X_F(F)$, one concludes that U meets the image of $X_F(F)$ in $X(\mathbb{A}_F^S)$ and the theorem is proved. \square

Corollary 2.11. *VSAP-BM off S holds for X_F if and only if the diagonal map*

$$\mathbf{X}(\mathfrak{o}_S) \rightarrow P_S[\mathbf{X}(\mathbb{A}_{F,S})^{\text{Br}(X_F)}]$$

is bijective for every model \mathbf{X} of X_F over \mathfrak{o}_S .

Proof. (\Rightarrow) We only have to show that the diagonal map is surjective. Let

$$x \in \mathbf{X}(\mathbb{A}_{F,S})^{\mathrm{Br}(X_F)} \subseteq X_F(\mathbb{A}_F)^{\mathrm{Br}(X_F)}.$$

Then there exists $\sigma \in X_F(F)$ such that for all $v \notin S$,

$$\sigma_v = x_v \in \mathbf{X}(\mathfrak{o}_v) \cap X_F(F) = \mathbf{X}(\mathfrak{o}_{F,v}),$$

where $\mathfrak{o}_{F,v}$ is the local ring of F at v . Hence $\sigma \in \mathbf{X}(\mathbb{A}_{F,S}) \cap X_F(F) = \mathbf{X}(\mathfrak{o}_S)$ and $\sigma = P_S(x)$ under the diagonal map.

(\Leftarrow) This follows from Corollary 2.9. \square

3. VERY STRONG APPROXIMATION WITH BRAUER-MANIN OBSTRUCTION

A smooth curve X_F is called hyperbolic if it is the complement of a reduced separable effective divisor of degree $d \geq 0$ on a projective curve of genus g such that $2g - 2 + d > 0$. By Siegel's theorem for integral points and Faltings' theorem on Mordell's conjecture, $\mathbf{X}(\mathfrak{o}_S)$ is finite for any integral model \mathbf{X} of X_F over \mathfrak{o}_S . If X_F satisfies SAP-BM off S (Definition 1.1), then the diagonal map

$$\mathbf{X}(\mathfrak{o}_S) \rightarrow P_S[\mathbf{X}(\mathbb{A}_{F,S})^{\mathrm{Br}(X_F)}]$$

is bijective for any integral model \mathbf{X} of X_F over \mathfrak{o}_S . By Corollary 2.11, X_F satisfies the VSAP-BM off S . However, Harari and Voloch gave an explicit example at the end of [12] to explain that the strong approximation property with Brauer-Manin obstruction does not hold in general.

Let us first prove some general properties regarding the very strong approximation property. Recall that an immersion is a closed immersion followed by an open immersion ([8], I.4.2.1).

Proposition 3.1. *Let $f : X_F \rightarrow Y_F$ be an immersion of algebraic varieties over a global field F . Suppose that VSAP-BM off S holds for Y_F , then the same holds for X_F .*

Proof. We have the following commutative diagram

$$\begin{array}{ccc} X_F(F) & \xrightarrow{f} & Y_F(F) \\ \downarrow & & \downarrow \\ X_F(\mathbb{A}_F)^{\mathrm{Br}(X_F)} & \longrightarrow & Y_F(\mathbb{A}_F)^{\mathrm{Br}(Y_F)} \subseteq Y_F(\mathbb{A}_F) \\ P_S \downarrow & & \downarrow P_S \\ P_S[X_F(\mathbb{A}_F)^{\mathrm{Br}(X_F)}] & \longrightarrow & P_S[Y_F(\mathbb{A}_F)^{\mathrm{Br}(Y_F)}] \subseteq Y_F(\mathbb{A}_F^S) \end{array}$$

where the horizontal arrows are injective (the last two ones are injective because $X_F(\mathbb{A}_F) \rightarrow Y_F(\mathbb{A}_F)$ and $X_F(\mathbb{A}_F^S) \rightarrow Y_F(\mathbb{A}_F^S)$ are injective). Here we use the hypothesis that f is an immersion (or at least universally injective). Moreover, if we consider $X_F(F)$, $Y_F(F)$ and $X_F(\mathbb{A}_F^S)$ as subsets of $Y_F(\mathbb{A}_F^S)$, then $Y_F(F) \cap X_F(\mathbb{A}_F^S) = X_F(F)$. Indeed, let $y \in Y_F(F)$ be such that $(y)_v = x_v$ in $Y_F(F_v)$ with $x_v \in X_F(F_v)$ for all $v \notin S$. Fix a $v_0 \notin S$, then $(y)_{v_0} \in X_F(F_{v_0}) \cap Y_F(F) = X_F(F)$. So $y \in X_F(F)$. In a more algebraic setting, we are saying that some F -algebra homomorphism $\mathcal{O}_{Y_F, y} \rightarrow F$

fits into a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{Y_F, y} & \xrightarrow{y} & F \\ \downarrow & & \downarrow \\ \mathcal{O}_{X_F, x_{v_0}} & \xrightarrow{x_{v_0}} & F_v \end{array}$$

where the left vertical arrow is surjective (here we use the hypothesis f is an immersion). So the bottom arrow takes values in F and $x_{v_0} \in X_F(F)$. This implies immediately the proposition. \square

Proposition 3.2. *Let X_F, Y_F be algebraic varieties over a global field F such that $X_F(F), Y_F(F) \neq \emptyset$. Then $X_F \times_F Y_F$ satisfies SAP-BM (resp. VSAP-BM) off S if and only if the same holds for both X_F and Y_F .*

Proof. By functoriality, the projections to X_F and Y_F induce a continuous map

$$(X_F \times_F Y_F)(\mathbb{A}_F)^{\text{Br}(X_F \times_F Y_F)} \rightarrow X_F(\mathbb{A}_F)^{\text{Br}(X_F)} \times Y_F(\mathbb{A}_F)^{\text{Br}(Y_F)}$$

which is injective as both sides are subsets of $X_F(\mathbb{A}_F) \times Y_F(\mathbb{A}_F)$. They both contain $(X_F \times_F Y_F)(F)$. The same holds when we project to $X_F(\mathbb{A}_F^S) \times Y_F(\mathbb{A}_F^S)$. So if X_F and Y_F satisfy VSAP-BM off S , then so does $X_F \times_F Y_F$. Suppose X_F, Y_F satisfy SAP-BM off S . As $(X_F \times_F Y_F)(\mathbb{A}_F)$ is canonically homeomorphic to the product $X_F(\mathbb{A}_F) \times Y_F(\mathbb{A}_F)$, any non-empty open subset of $P_S[(X_F \times_F Y_F)(\mathbb{A}_F)^{\text{Br}(X_F \times_F Y_F)}]$ is the trace of some $U \times V$ with U, V respective open subsets of $X_F(\mathbb{A}_F^S)$ and $Y_F(\mathbb{A}_F^S)$ such that $U \cap P_S(X_F(\mathbb{A}_F)^{\text{Br}(X_F)})$ and $V \cap P_S(Y_F(\mathbb{A}_F)^{\text{Br}(Y_F)})$ are non-empty. Thus we find rational points $x \in X_F(F) \cap U$ and $y \in Y_F(F) \cap V$, therefore $(x, y) \in (X_F \times_F Y_F)(F) \cap (U \times V)$ and $X_F \times_F Y_F$ satisfies SAP-BM off S .

Now we notice that a rational point $y_0 \in Y_F(F)$ gives rise to a closed immersion

$$s_{y_0} : X_F \rightarrow X_F \times_F Y_F, \quad x \mapsto (x, y_0)$$

by base change. Thus X_F (and similarly, Y_F) satisfies VSAP-BM off S if $X_F \times_F Y_F$ does (Proposition 3.1). Suppose now that $X_F \times_F Y_F$ satisfies SAP-BM off S . Let U be a non-empty subset of $P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}]$. The intersection

$$(U \times P_S[Y_F(\mathbb{A}_F)^{\text{Br}(Y_F)}]) \cap P_S[(X_F \times_F Y_F)(\mathbb{A}_F)^{\text{Br}(X_F \times_F Y_F)}]$$

is non-empty as it contains $U \times \{y_0\}$, thus contains a point $(x, y) \in (X_F \times_F Y_F)(F)$. By construction, $x \in U \cap X_F(F)$, and X_F satisfies SAP-BM off S . \square

Note that for projective smooth and geometrically integral varieties, Skorobogatov and Zarhin proved a stronger result ([22], Theorem C) than Proposition 3.2.

The simplest class for which the very strong approximation property with Brauer-Manin obstruction holds are zero-dimensional schemes over F .

Proposition 3.3. *Let Ξ be the set of the complex places of F if F is a number field and $\Xi = \emptyset$ otherwise. If $X_F = X_1 \coprod X_2$ is a disjoint union of two open subschemes, then*

$$P_\Xi[X_F(\mathbb{A}_F)^{\text{Br}(X)}] = P_\Xi[X_1(\mathbb{A}_F)^{\text{Br}(X_1)}] \coprod P_\Xi[X_2(\mathbb{A}_F)^{\text{Br}(X_2)}].$$

In particular, if X_F is finite, then X_F satisfies the very strong approximation property with Brauer-Manin obstruction off any finite set containing Ξ .

Proof. If there is

$$(x_v)_{v \in (\Omega_F \setminus \Xi)} \in P_\Xi[X_F(\mathbb{A}_F)^{\text{Br}(X)}]$$

with $v_1, v_2 \in \Omega_F \setminus \Xi$ such that $x_{v_1} \in X_1(F_{v_1})$ and $x_{v_2} \in X_2(F_{v_2})$, then one can choose $\xi_1 \in \text{Br}(F)$ and $\xi_2 \in \text{Br}(F)$ satisfying

$$\begin{cases} \text{inv}_v(\xi_1) \neq \text{inv}_v(\xi_2) & \text{if } v = v_1 \text{ or } v_2 \\ \text{inv}_v(\xi_1) = \text{inv}_v(\xi_2) = 0 & \text{otherwise} \end{cases}$$

by the Artin reciprocity law

$$0 \rightarrow \text{Br}(F) \rightarrow \bigoplus_{v \in \Omega_F} \text{Br}(F_v) \xrightarrow{\sum_v \text{inv}_v} \mathbb{Q}/\mathbb{Z} \rightarrow 0$$

(see [17], VIII, (8.1.17)).

Let ξ_i^* be the images of ξ_i under the canonical map $\text{Br}(F) \rightarrow \text{Br}(X_i)$ for $i = 1, 2$. Then

$$(\xi_1^*, \xi_2^*) \in \text{Br}(X_1) \oplus \text{Br}(X_2) \cong \text{Br}(X)$$

and

$$0 = \sum_{v \in (\Omega_F \setminus \Xi)} \text{inv}_v((\xi_1^* + \xi_2^*)(x_v)) = \text{inv}_{v_1}(\xi_1) + \text{inv}_{v_2}(\xi_2) = \text{inv}_{v_1}(\xi_1) - \text{inv}_{v_1}(\xi_2)$$

by the Artin reciprocity law as above. A contradiction is derived and the first part of the proposition is proved.

Now suppose that X_F is finite. By the above result, we can suppose X_F is local. By [9], Cor. 6.2, we can further suppose that X_F is reduced, hence is the spectrum of some finite field extension L of F . By Chebotarev Density Theorem, if $X_F(\mathbb{A}_F) \neq \emptyset$, then $L = F$, in which case $X_F(F) = X_F(\mathbb{A}_F)$ and the result follows. \square

Note that a similar decomposition result can be found in [23], Proposition 5.1.1.

Let T_F be a torus over a global field F . Let $\widehat{T}_F := \text{Hom}_{\bar{F}}(T_{\bar{F}}, \mathbb{G}_{m, \bar{F}})$ be the group of characters of T_F . A point $x \in T_F(\mathbb{A}_F)$ induces an evaluation homomorphism

$$\widehat{T}_F \xrightarrow{x} \mathbb{I}_{\bar{F}} := \varinjlim_{E/F} \mathbb{I}_E$$

where E/F runs over all finite separable subextensions of \bar{F} , thus a group homomorphism

$$H^2(F, \widehat{T}_F) \xrightarrow{x^*} H^2(F, \mathbb{I}_{\bar{F}}) = \bigoplus_{v \in \Omega_F} \text{Br}(F_v) \xrightarrow{\sum_v \text{inv}_v} \mathbb{Q}/\mathbb{Z}$$

([17], VIII, (8.1.7)). This gives a pairing

$$T_F(\mathbb{A}_F) \times H^2(F, \widehat{T}_F) \longrightarrow \mathbb{Q}/\mathbb{Z} \quad (3.1)$$

and induces a homomorphism

$$T_F(\mathbb{A}_F) \longrightarrow \text{Hom}(H^2(F, \widehat{T}_F), \mathbb{Q}/\mathbb{Z}). \quad (3.2)$$

This pairing is compatible with the Brauer-Manin pairing. Indeed, one has the canonical inclusion

$$\text{Br}(F) \oplus H^2(F, \widehat{T}_F) = H^2(F, \mathcal{O}(T_{\bar{F}})^\times) \hookrightarrow \text{Br}(T_F)$$

by Hochschild-Serre spectral sequence (see [15], III, Theorem 2.20) with $\text{Pic}(T_{\bar{F}}) = 0$. This inclusion commutes with the evaluation

$$\begin{array}{ccc} H^2(F, \widehat{T}_F) & \longrightarrow & \text{Br}(T_F) \\ x^* \downarrow & & \downarrow x^* \\ \bigoplus_{v \in \Omega_F} \text{Br}(F_v) & \xlongequal{\quad} & \bigoplus_{v \in \Omega_F} \text{Br}(F_v) \end{array}$$

for any $x \in T_F(\mathbb{A}_F)$.

For any topological group G , denote by G^\wedge the completion of G with respect to open finite index subgroups.

Proposition 3.4. *Let T_F be a torus over a global function field F . Then we have canonical exact sequences*

$$1 \rightarrow T_F(F)^\wedge \rightarrow T_F(\mathbb{A}_F)^\wedge \rightarrow H^2(F, \widehat{T}_F)^\vee \quad (3.3)$$

and

$$1 \rightarrow T_F(F) \rightarrow T_F(\mathbb{A}_F) \rightarrow H^2(F, \widehat{T}_F)^\vee \quad (3.4)$$

where $^\vee$ is the Pontryagin dual.

Proof. The first exact sequence follows from Theorem 3.11 in [7] by taking $M = T$. Indeed, the hyper-cohomology is the same as the usual Galois cohomology in this case and $M^* = \widehat{T}[1]$. Since $H^3(F, \widehat{T})$ is torsion, one has $H^2(F, M^*) = H^3(F, \widehat{T}) = 0$ by VIII, (8.3.17) in [17].

The second one is deduced from (3.3) if we can prove that the canonical map

$$T_F(\mathbb{A}_F)/T_F(F) \rightarrow T_F(\mathbb{A}_F)^\wedge/T_F(F)^\wedge$$

is injective. Since $T_F(F)$ is discrete in $T_F(\mathbb{A}_F)$, the quotient group $T_F(\mathbb{A}_F)/T_F(F)$ is locally compact and totally disconnected. Let S be a finite non-empty subset of Ω_F and \mathbf{T} be a group scheme model of T over \mathfrak{o}_S . Then $T_F(\mathbb{A}_F)/T_F(F)\mathbf{T}(\mathbb{A}_{F,S})$ is finite (when T_F is split this comes from the finiteness of $\text{Pic}(\mathfrak{o}_S)$, the general case follows from [13], Lemma 2.2.3). It is clear that $T_F(F)\mathbf{T}(\mathbb{A}_{F,S})/T_F(F)$ is compactly generated. This implies that $T_F(\mathbb{A}_F)/T_F(F)$ is compactly generated. Hence

$$T_F(\mathbb{A}_F)/T_F(F) \hookrightarrow (T_F(\mathbb{A}_F)/T_F(F))^\wedge$$

(see e.g. [11], Appendix). Since the natural projection $T_F(\mathbb{A}_F) \rightarrow T_F(\mathbb{A}_F)/T_F(F)$ is open, one concludes

$$(T_F(\mathbb{A}_F)/T_F(F))^\wedge = T_F(\mathbb{A}_F)^\wedge/T_F(F)^\wedge$$

by Appendix in [11]. □

Theorem 3.5. *Let T_F be a torus over a global field F . Suppose that one of the following conditions holds:*

- (1) F is a global function field;¹
- (2) F is a number field and

$$\text{rank}_F(T_F) = \sum_{v \in \infty_F} \text{rank}_{F_v}(T_{F_v})$$

where $\text{rank}_F(T_F)$ and $\text{rank}_{F_v}(T_{F_v})$ are the respective ranks of the free abelian groups $\text{Hom}_F(T_F, \mathbb{G}_m)$ and $\text{Hom}_{F_v}(T_{F_v}, \mathbb{G}_m)$.

¹In this case $\infty_F = \emptyset$ and S can be empty, P_\emptyset is then the identity map

Then T_F satisfies VSAP-BM off S for any finite subset S containing ∞_F . In Case (2), $T_F(F)$ is discrete, hence closed in $T_F(\mathbb{A}_{\mathbb{Q}}^{\infty_F})$.

Proof. It is enough to show that, under the hypothesis of the theorem, the kernel of the map (3.2) is $T_F(F)$.

(1) The case where F is a global function field follows from Proposition 3.4.

(2) Suppose F is a number field. According to [20], Main Theorem, under the hypothesis on the ranks, there is an open subgroup of $T(\mathbb{A}_F)$ whose intersection with $T_F(F)$ is finite. So $T_F(F)$ is discrete (hence closed) in $T_F(\mathbb{A}_F^{\infty_F})$. The result follows from [10], Théorème 2 (see also Remarques that follow). \square

Recall that a subvariety is a closed subvariety of an open subvariety.

Corollary 3.6. *Let X_F be a subvariety of a torus as in Theorem 3.5. Then*

$$X_F(F) = P_{\infty_F}(X_F(\mathbb{A}_F)^{\text{Br}(X)})$$

where ∞_F is empty if F is a global function field. In particular, for any finite $S \supseteq \infty_F$, the very strong approximation property with Brauer-Manin obstruction off S holds for X_F .

Proof. This follows from Proposition 3.1 and Theorem 3.5. \square

Example 3.7 The following tori satisfy the rank condition in Theorem 3.5 (2):

- (1) when $F = \mathbb{Q}$ or is a quadratic imaginary field and T_F is a split torus over F ;
- (2) when F is a totally real field, and $T_F = \prod_{i=1}^d \text{Res}_{L_i/F}^1(\mathbb{G}_{m,L_i})$ where the L_i are totally imaginary quadratic extensions of F and $\text{Res}_{L_i/F}$ denotes the Weil restriction functor;
- (3) when T_F is isogeneous to any finite product of the above tori over F .

Example 3.8 Let $X_{\mathbb{Q}}$ be one of the algebraic varieties defined as follows. Then VSAP-BM off $\infty_{\mathbb{Q}}$ holds for $X_{\mathbb{Q}}$.

- (1) Let $X_{\mathbb{Q}}$ be defined by the following diagonal equation

$$\sum_{i=1}^d a_i x_i^{n_i} = a_0 \quad \text{with} \quad \prod_{i=1}^d x_i \neq 0$$

where $a_i \in \mathbb{Q}$, $n_i \in \mathbb{N} \setminus \{0\}$ for $1 \leq i \leq d$. Then $X_{\mathbb{Q}}$ is a closed subvariety of $\mathbb{G}_{m,\mathbb{Q}}^d$.

- (2) Let E be an elliptic curve over \mathbb{Q} and $z_1, z_2, z_3 \in E(\mathbb{Q})[3]$ be three distinct points. Then

$$X_{\mathbb{Q}} = E \setminus \{z_1, z_2, z_3\}$$

can be embedded in $\mathbb{G}_{m,\mathbb{Q}}^2$ because it is contained in the complementary in $\mathbf{P}_{\mathbb{Q}}^2$ of the tangent lines at the three missing points.

- (3) Any open subset of $\mathbf{P}_{\mathbb{Q}}^1$ in the complementary of an imaginary quadratic point is contained in a torus as in Example 3.7(2).

Remark 3.9 Suppose X_F is a smooth and geometrically integral variety over F and U is an open subscheme of X_F . If $\text{Br}(U)/\text{Br}(F)$ is finite, then the strong approximation property with Brauer-Manin obstruction of U implies the strong approximation property with Brauer-Manin obstruction of X_F by Proposition 2.6 in [4]. It is natural to ask if the finiteness condition for $\text{Br}(U)/\text{Br}(F)$ can be removed.

Take any X_F for which the strong approximation property with Brauer-Manin obstruction does not hold (*e.g.*, the complement of a rational point in some elliptic curve as at the end of [12]), then the next corollary (or Example 3.8(2)) gives a negative answer to the above question, by taking a small enough open subset U of X_F .

Corollary 3.10. *Let X_F be an algebraic variety over F , where $F = \mathbb{Q}$ or is an imaginary quadratic field. Then there exists a dense open subset U of X_F such that U satisfies the very strong approximation property with Brauer-Manin obstruction off S for any finite subset S containing ∞_F .*

Proof. By Corollary 3.6 and Example 3.7(1), it is enough to show that X_F contains a dense open subset U which is a finite disjoint union of subvarieties of split tori over F . After restricting to the interior of each irreducible component, we can suppose that X_F is irreducible and even integral. Shrinking further X_F if necessary, we can suppose that X_F is affine. Now embed X_F as a closed subvariety in some affine space \mathbb{A}_F^n with n minimal. Let T be a split torus obtained as the complement of n independent linear hyperplanes in \mathbb{A}_F^n . Then $U := X_F \cap T$ is a closed subvariety of T , is non-empty (by the minimality of n) and is open in X_F . \square

Example 3.11 Let F be a number field. Let X_F be an algebraic variety admitting a non-constant morphism $f : \mathbf{A}_F^1 \rightarrow X_F$ from the affine line over F . Then for any finite $S \supseteq \infty_F$, the map

$$X_F(F) \rightarrow P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}]$$

is not surjective. This can be proved as follows. The image of f in X_F is a locally closed subset of dimension 1. Endowed with the reduced structure, it is an integral subvariety of X_F . By Proposition 3.1, we can replace X_F by $f(\mathbf{A}_F^1)$ and suppose that f is surjective.

We can extend f to a finite surjective morphism $\pi : \mathbf{P}_F^1 \rightarrow \overline{X}_F$ to a compactification \overline{X}_F of X_F . Then $\pi^{-1}(X_F) = \mathbf{A}_F^1$. By Lüroth's Theorem, the normalization of X_F is isomorphic to \mathbf{A}_F^1 . Thus we can replace f by the normalization map of X_F and suppose that f is birational. Let V be the smooth locus of X_F and let $U = f^{-1}(V)$. The canonical injective map

$$P_S[U(\mathbb{A}_F)] \cong P_S[V(\mathbb{A}_F)] \rightarrow P_S[X_F(\mathbb{A}_F)]$$

factors through

$$P_S[U(\mathbb{A}_F)] \subseteq P_S[\mathbf{A}_F^1(\mathbb{A}_F)] = P_S[\mathbf{A}_F^1(\mathbb{A}_F)^{\text{Br}(\mathbf{A}_F^1)}] \rightarrow P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}].$$

So $P_S[U(\mathbb{A}_F)]$ is actually contained in $P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}]$. If the map

$$X_F(F) \rightarrow P_S[X_F(\mathbb{A}_F)^{\text{Br}(X_F)}]$$

is surjective, as $X_F(F)$ is countable, $P_S[U(\mathbb{A}_F)]$ would also be countable, we have a contradiction.

4. HARARI-VOLOCH'S CONJECTURE

Let X_F be a hyperbolic rational curve (*i.e.* X_F is an open subset in \mathbf{P}_F^1 , and its complementary is a reduced separable divisor of degree ≥ 3) and let \mathbf{X} be an integral model of X_F over \mathfrak{o}_S for a non-empty finite subset S of Ω_F containing ∞_F . Harari and Voloch proposed the following conjecture ([12], Conjecture 2).

Conjecture 4.1. (*Harari-Voloch*) *Let*

$$B_S(X_F) := \text{Ker}[\text{Br}(X_F) \rightarrow \prod_{v \in S} \text{Br}(X_{F_v}) / \text{Br}(F_v)].$$

Then the diagonal map

$$\mathbf{X}(\mathfrak{o}_S) \rightarrow \left(\prod_{v \notin S} \mathbf{X}(\mathfrak{o}_v) \right)^{B_S(X_F)}$$

is bijective.

Remark 4.2 By the same argument as those in Corollary 2.11, this conjecture holds for all models of X_F over $\text{Spec}(\mathfrak{o}_S)$ if and only if

$$X_F(F) \rightarrow X_F(\mathbb{A}_F^S)^{B_S(X_F)}$$

is bijective. Since

$$X_F(\mathbb{A}_F^S)^{B_S(X_F)} \supseteq P_S[(X_F(\mathbb{A}_F))^{\text{Br}(X_F)}] \supseteq X_F(F),$$

Conjecture 4.1 implies the very strong approximation property with Brauer-Manin obstruction off S for X_F , and it also implies a conjecture of Skolem (see Remark 2.5 in [12]).

Lemma 4.3. *Let F be any field. Let X_F be an affine curve contained in \mathbf{P}_F^1 with $X_F(F) \neq \emptyset$. Let \bar{F} be a separable closure of F .*

(1) *There is a split exact sequence*

$$0 \rightarrow \text{Br}(F) \rightarrow \text{Br}(X_F) \rightarrow H^2(F, \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times) \rightarrow 0.$$

(2) *Suppose F is a global field. Denote by \bar{F}_v a separable closure of F_v for any $v \in S$. Then one has a split exact sequence*

$$0 \rightarrow \text{Br}(F) \rightarrow B_S(X_F) \rightarrow$$

$$\text{Ker} \left[H^2(F, \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times) \xrightarrow{\text{res}} \prod_{v \in S} H^2(F_v, \mathcal{O}(X_{\bar{F}_v})^\times / \bar{F}_v^\times) \right] \rightarrow 0 \quad (4.1)$$

for any finite subset S of Ω_F containing ∞_F .

Proof. By Hochschild-Serre spectral sequence (see [15], III, Theorem 2.20 and Appendix B, p. 309)

$$H^p(G, H^q(X_{\bar{F}}, \mathbb{G}_m)) \Rightarrow H^{p+q}(X_F, \mathbb{G}_m)$$

with $G = \text{Gal}(\bar{F}/F)$, one has the long exact sequence

$$(\text{Pic}(X_{\bar{F}}))^G \rightarrow H^2(F, \mathcal{O}(X_{\bar{F}})^\times) \rightarrow \text{Ker}[\text{Br}(X_F) \rightarrow \text{Br}(X_{\bar{F}})] \rightarrow H^1(F, \text{Pic}(X_{\bar{F}})).$$

Since X_F is an affine open subscheme of \mathbf{P}_F^1 , one has $\text{Pic}(X_{\bar{F}}) = 0$. Moreover, $\text{Br}(X_{\bar{F}}) \subseteq \text{Br}(\bar{F}(X_{\bar{F}})) = 0$ by Tsen's Theorem. Therefore

$$\text{Br}(X_F) \cong H^2(F, \mathcal{O}(X_{\bar{F}})^\times).$$

The existence of a F -rational point on X_F implies that the exact sequence of G -modules

$$1 \rightarrow \bar{F}^\times \rightarrow \mathcal{O}(X_{\bar{F}})^\times \rightarrow \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times \rightarrow 1$$

splits. This give a split exact sequence

$$0 \rightarrow \text{Br}(F) \rightarrow \text{Br}(X_F) \rightarrow H^2(F, \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times) \rightarrow 0.$$

When F is a global field, the same decomposition holds over F_v for any $v \in S$, and the lemma follows easily. \square

Let F be any field and X_F be an affine curve contained in \mathbf{P}_F^1 such that the reduced divisor $D := \mathbf{P}_F^1 \setminus X_F$ is separable of degree ≥ 2 and $X_F(F) \neq \emptyset$. The next lemma is the same as part of [12], Lemma 2.1, but we do not assume $H^3(F, \mathbb{G}_m) = 0$.

Lemma 4.4. *Let X_F be as above and let \bar{F} be a separable closure of F with Galois group G . Let $\bar{D} = D \times_F \bar{F}$ and denote by $\text{Div}_{\bar{D}}(\mathbf{P}_{\bar{F}}^1)$ the group of divisors on $\mathbf{P}_{\bar{F}}^1$ supported in \bar{D} . Let T_F be a torus over F such that its character group \hat{T}_F is isomorphic to $\text{Div}_{\bar{D}}^0(\mathbf{P}_{\bar{F}}^1)$ as a G -module. Then there exists a closed immersion*

$$j : X_F \rightarrow T_F$$

such that the induced group homomorphism

$$\mathcal{O}(T_{\bar{F}})^\times / \bar{F}^\times \rightarrow \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times$$

is an isomorphism.

Proof. We have a canonical exact sequence

$$1 \rightarrow \bar{F}^\times \rightarrow \mathcal{O}(X_{\bar{F}})^\times \rightarrow \text{Div}_{\bar{D}}(\mathbf{P}_{\bar{F}}^1)$$

where the last map consists in taking the divisors of rational functions on $X_{\bar{F}}$. Because $\text{Pic}^0(\mathbf{P}_{\bar{F}}^1) = 1$, it is clear that the above exact sequence induces an exact sequence of G -modules

$$1 \rightarrow \bar{F}^\times \rightarrow \mathcal{O}(X_{\bar{F}})^\times \rightarrow \text{Div}_{\bar{D}}^0(\mathbf{P}_{\bar{F}}^1) \rightarrow 0.$$

A rational point on X_F gives rise to an evaluation homomorphism $e : \mathcal{O}(X_{\bar{F}})^\times \rightarrow \bar{F}^\times$ of G -modules, which is identity on \bar{F}^\times on the source. Thus we have G -equivariant isomorphisms

$$\text{Div}_{\bar{D}}^0(\mathbf{P}_{\bar{F}}^1) \cong \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times \cong e^{-1}(\{1\}) \subset \mathcal{O}(X_{\bar{F}})^\times,$$

and a G -equivariant \bar{F} -algebras homomorphism

$$\bar{F}[\text{Div}_{\bar{D}}^0(\mathbf{P}_{\bar{F}}^1)] \rightarrow \bar{F}[e^{-1}(\{1\})] = \bar{F}[\mathcal{O}(X_{\bar{F}})^\times] \subseteq \mathcal{O}(X_{\bar{F}}).$$

An easy explicit computation (using the coordinates of the points of \bar{D} ; the hypothesis $\deg D \geq 2$ is needed here) shows that the last two terms are actually equal. Then the above homomorphism induces a closed immersion $X_F \rightarrow T_F$ (depending on the choice of a rational point on X_F), and an isomorphism

$$\mathcal{O}(T_{\bar{F}})^\times / \bar{F}^\times = \text{Div}_{\bar{D}}^0(\mathbf{P}_{\bar{F}}^1) \rightarrow \mathcal{O}(X_{\bar{F}})^\times / \bar{F}^\times.$$

□

Definition 4.5 Let T_F be a torus over F and let \hat{T}_F be the group of characters of T_F . When F is a global field, we set

$$B_{1,S}(T_F) = \text{Ker}(H^2(F, \hat{T}_F) \rightarrow \prod_{v \in S} H^2(F_v, \hat{T}_F)).$$

For any field F , we have $\hat{T}_F \cong \mathcal{O}(T_{\bar{F}})^\times / \bar{F}^\times$ as Galois modules. Similarly to the proof of Lemma 4.3(1), we have a split exact sequence

$$0 \rightarrow \text{Br}(F) \rightarrow \text{Br}_1(T_F) := \text{Ker}(\text{Br}(T_F) \rightarrow \text{Br}(T_{\bar{F}})) \rightarrow H^2(F, \hat{T}_F) \rightarrow 0.$$

The following proposition is a variant of Theorem 1 in [12].

Proposition 4.6. *Let X_F be a hyperbolic rational curve over a global field F and let $j : X_F \rightarrow T_F$ be the closed immersion given by Lemma 4.4. Fix a non-empty finite set S of primes containing ∞_F . Then*

(1) we have

$$\overline{T_F(F)}^S = T_F(\mathbb{A}_F^S)^{B_{1,S}(T_F)}$$

where $\overline{T_F(F)}^S$ is the topological closure of $T_F(F)$ inside $T_F(\mathbb{A}_F^S)$;

(2) Conjecture 4.1 for all models \mathbf{X} of X_F over \mathfrak{o}_S is true if and only if

$$\overline{T_F(F)}^S \cap X_F(\mathbb{A}_F^S) = X_F(F).$$

Proof. By Lemmas 4.3 and 4.4, we have a split exact sequence:

$$0 \rightarrow \mathrm{Br}(F) \rightarrow B_S(X_F) \rightarrow B_{1,S}(T_F) \rightarrow 0. \quad (4.2)$$

By Theorem 2 in [10] when F is a number field, and by Proposition 3.4 when F is a function field, together with the compatibility of local duality ([17], VII, (7.2.9)), one has the following diagram of exact sequences

$$\begin{array}{ccccc} (\prod_{v \in \infty_F} \pi_0(T_F(F_v))) \times \prod_{v \in S \setminus \infty_F} T_F(F_v) & \xrightarrow{\phi} & \prod_{v \in S} H^2(F_v, \widehat{T}_F)^\vee \\ \downarrow & & \downarrow \mathrm{res}^\vee \\ \overline{T_F(F)} \longrightarrow (\prod_{v \in \infty_F} \pi_0(T_F(F_v))) \times T_F(\mathbb{A}_F^{\infty_F}) & \xrightarrow{\rho} & H^2(F, \widehat{T}_F)^\vee \\ & \downarrow P_{0,S} & \downarrow \\ & T_F(\mathbb{A}_F^S) & \longrightarrow B_{1,S}(T_F)^\vee \end{array}$$

where $^\vee$ is the Pontryagin dual, $\overline{T_F(F)}$ is the topological closure of $T_F(F)$ inside $(\prod_{v \in \infty_F} \pi_0(T_F(F_v))) \times T_F(\mathbb{A}_F^{\infty_F})$, $\pi_0(T_F(F_v))$ is the group of connected components of the topological group $T_F(F_v)$ and $P_{0,S}$ is the projection map. Moreover, ϕ is injective with dense image by [17], VII, (7.2.10). Write for simplicity this diagram as

$$\begin{array}{ccccc} T_S & \xrightarrow{\phi} & H_S^\vee \\ \downarrow & & \downarrow \mathrm{res}^\vee \\ \overline{T_F(F)} \longrightarrow G & \xrightarrow{\rho} & H^\vee \\ \downarrow P_{0,S} & & \downarrow \\ T_F(\mathbb{A}_F^S) & \longrightarrow & B_S^\vee \end{array}$$

and denote $T_0^S = T_F(\mathbb{A}_F^S)^{B_{1,S}(T_F)}$. As the latter is closed in $T_F(\mathbb{A}_F^S)$, to prove (1), one only needs to show that $T_F(F)T_S$ is dense in $P_{0,S}^{-1}(T_0^S)$ or, equivalently, that $\overline{T_F(F)}T_S$ is dense in $P_{0,S}^{-1}(T_0^S)$.

When F is a number field, the image of ρ is the kernel of a continuous map of topological groups $H^2(F, \widehat{T}_F)^\vee \rightarrow \mathrm{III}^1(T_F)$ by [10], Theorem 2, hence compact. Because its domain is σ -compact, ρ is an open map to its image. Let $U \subseteq G$ be an open subset such that

$$U \cap P_{0,S}^{-1}(T_0^S) \neq \emptyset.$$

Then $\rho(U) = W \cap \rho(G)$ for some open subset W of H^\vee . As $(\mathrm{res}^\vee)^{-1}(W)$ is a non-empty open subset of H_S^\vee by the hypothesis on U , it contains an element of $\phi(T_S)$ by the density of $\phi(T_S)$. This implies that U contains an element of $(\mathrm{Ker} \rho)T_S = \overline{T_F(F)}T_S$ and proves (1) for number fields.

Now suppose that F is a global function field. Then

$$\overline{T_F(F)} = T_F(F) \quad \text{and} \quad G = T_F(\mathbb{A}_F).$$

Let U be an open subset of $T_F(\mathbb{A}_F)$ with an intersection point $g \in U \cap P_{0,S}^{-1}(T_0^S)$. Let C be the integral smooth projective curve whose function field is F . Then $g^{-1} \cdot U$ contains an open subgroup of the form $\prod_v \mathbf{T}(\mathfrak{o}_v)$, where \mathbf{T} is a group scheme of finite type over C with generic fiber isomorphic to T_F (Corollary 2.9). We can shrink U and suppose $U = g \prod_v \mathbf{T}(\mathfrak{o}_v)$. As $T_F(F)T_S$ is a group, $U \cap T_F(F)T_S \neq \emptyset$ is equivalent to $1 \in gH$, where

$$H := \left(\prod_v \mathbf{T}(\mathfrak{o}_v) \right) T_F(F)T_S = T_F(F) \left(\prod_{v \in S} T_F(F_v) \times \prod_{v \notin S} \mathbf{T}(\mathfrak{o}_v) \right)$$

is an open subgroup of $T_F(\mathbb{A}_F)$. We know that $T_F(\mathbb{A}_F)/H$ is finite (when T_F is split this comes from the finiteness of $\text{Pic}(\mathfrak{o}_S)$, the general case follows from [13], Lemma 2.2.3). Therefore, H is an open subgroup of finite index in $T_F(\mathbb{A}_F)$ and gH^\wedge is an open subset in $T_F(\mathbb{A}_F)^\wedge$. Similarly to the case of number fields, by using the exact sequence (3.3), we see that $gH^\wedge \cap T_F(F)^\wedge T_S \neq \emptyset$. As $T_F(F)$ and T_S are contained in H , $T_F(F)^\wedge T_S \subseteq H^\wedge$, hence $g \in H^\wedge \cap T_F(\mathbb{A}_F) = H$ and $1 \in gH$.

(2) By (1), we have

$$\overline{T_F(F)}^S \cap X_F(\mathbb{A}_F^S) = (T_F(\mathbb{A}_F^S))^{B_{1,S}(T_F)} \cap X_F(\mathbb{A}_F^S) = X_F(\mathbb{A}_F^S)^{B_S(X_F)}$$

by the functoriality of the Brauer-Manin pairing (see [21], (5.3)) and by the exact sequence (4.2). This completes the proof of (2) by Remark 4.2. \square

Corollary 4.7. *Conjecture 4.1 is true if F is a global function field.*

Proof. Embed X_F into a torus T_F as in Lemma 4.4. By the above proposition, it is enough to show that $\overline{T_F(F)}^S \cap X_F(\mathbb{A}_F^S) \subseteq X_F(F)$. For any $x \in \overline{T_F(F)}^S \cap X_F(\mathbb{A}_F^S)$, there is a finite set $S_1 \supseteq S$ such that T_F can be extended to a group scheme \mathbf{T} of finite type over \mathfrak{o}_{S_1} with

$$x \in \prod_{v \in S_1 \setminus S} T_F(F_v) \times \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v).$$

The latter is an open subset whose intersection with $T_F(F)$ is equal to $\mathbf{T}(\mathfrak{o}_{S_1})$. This implies that $x \in \overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ where $\overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ is the topological closure of $\mathbf{T}(\mathfrak{o}_{S_1})$ in $\prod_{v \notin S} T_F(F_v)$ with the product topology. Since $\mathbf{T}(\mathfrak{o}_{S_1})$ is a finitely generated abelian group, one has

$$x \in \overline{\mathbf{T}(\mathfrak{o}_{S_1})} \cap X_F(\mathbb{A}_F^S) = \mathbf{T}(\mathfrak{o}_{S_1}) \cap X_F(F) \subseteq X_F(F)$$

by [24], Theorem 1. The proof is complete. \square

Example 4.8 (See also Corollary 5.11) Let us give some evidence for Harari-Voloch conjecture over \mathbb{Q} with $S = \{\infty\}$. Let $X_{\mathbb{Q}}$ be a non-empty open subset of $\mathbf{P}_{\mathbb{Q}}^1 \setminus D$, where D consists of either three rational points, or two imaginary quadratic points, or one point of each type. Then Harari-Voloch conjecture is true for $X_{\mathbb{Q}}$ and $S = \{\infty\}$.

Indeed, the torus $T_{\mathbb{Q}}$ which the curve $Y_{\mathbb{Q}} := \mathbf{P}_{\mathbb{Q}}^1 \setminus D$ is embedded in following Lemma 4.4 satisfies the condition of Theorem 3.5 (2). In particular,

$$\overline{T_{\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}})}^S = T_{\mathbb{Q}}(\mathbb{Q}).$$

It follows from Proposition 4.6 (2) that Harari-Voloch conjecture holds for $Y_{\mathbb{Q}}$ and S . For arbitrary non-empty open subset $X_{\mathbb{Q}}$ of $Y_{\mathbb{Q}}$, this proof does not work anymore, but the result is still true by applying [12], Theorem 3 to the open immersion $X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}}$.

5. ADELIC POINTS OF ZERO-DIMENSIONAL SUBVARIETIES OF TORI

In this section we will show that the analogue of [23], Theorem 3.11 holds for tori over number fields. See Theorem 5.7. As an application, we generalize [12], Theorem 3. See Proposition 5.9. Throughout this section, F is a number field. We also fix a finite subset S of Ω_F containing ∞_F .

The proof of the main theorem 5.7 of this section follows closely the same strategy as in [23]. Let us first show some preliminary results on Galois cohomology of tori. Let S_1 be a finite subset of Ω_F with $S_1 \supseteq S$ and \mathbf{T} be a separated commutative group scheme of finite type over \mathfrak{o}_{S_1} such that

$$\mathbf{T} \times_{\mathfrak{o}_{S_1}} \bar{\mathfrak{o}}_{S_1} \cong \mathbb{G}_{m, \bar{\mathfrak{o}}_{S_1}}^d \quad (5.1)$$

for some $d \geq 1$, where $\bar{\mathfrak{o}}_{S_1}$ is the integral closure of \mathfrak{o}_{S_1} inside the algebraic closure \bar{F} of F . Under the assumption (5.1), the multiplication by N on $\mathbf{T}(\bar{\mathfrak{o}}_{S_1}) = (\bar{\mathfrak{o}}_{S_1}^*)^d$ is surjective. Note that enlarging S_1 if necessary, Condition (5.1) is always satisfied. Indeed, we can extend F and suppose $\mathbf{T}_F \simeq \mathbb{G}_{m, F}^d$. As \mathbf{T} and $\mathbb{G}_{m, \mathfrak{o}_{S_1}}^d$ coincide on the generic fiber and are both of finite type, they coincide on a dense open subset of $\text{Spec } \mathfrak{o}_{S_1}$.

For any positive integer N , denote by $\mathbf{T}[N]$ the group scheme of N -torsion of \mathbf{T} . In what follows, in Galois cohomology groups, \mathbf{T} (resp. $\mathbf{T}[N]$) is the Galois module $\mathbf{T}(\bar{\mathfrak{o}}_{S_1})$ (resp. $\mathbf{T}[N](\bar{\mathfrak{o}}_{S_1})$). Define as usual

$$\text{Sel}_S^{(N)}(F, \mathbf{T}) := \text{Ker}(H^1(F, \mathbf{T}[N]) \rightarrow \prod_{v \notin S} H^1(F_v, \mathbf{T}))$$

and

$$\text{III}_S(\mathbf{T}) := \text{Ker}(H^1(F, \mathbf{T}) \rightarrow \prod_{v \notin S} H^1(F_v, \mathbf{T})).$$

(It can be seen that the image of $H^1(F, \mathbf{T})$ in $\prod_{v \notin S} H^1(F_v, \mathbf{T})$ is in fact contained in $\bigoplus_{v \notin S} H^1(F_v, \mathbf{T})$ by Lang's Theorem on torsors under connected algebraic groups over a finite field). Then one has the short exact sequence

$$0 \rightarrow \mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}) \rightarrow \text{Sel}_S^{(N)}(F, \mathbf{T}) \rightarrow \text{III}_S(\mathbf{T})[N] \rightarrow 0 \quad (5.2)$$

and the natural coordinate map can be decomposed as

$$\mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}) \rightarrow \text{Sel}_S^{(N)}(F, \mathbf{T}) \rightarrow \left(\prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v)/N\mathbf{T}(\mathfrak{o}_v) \right) \times \prod_{v \in S_1 \setminus S} T_F(F_v)/NT_F(F_v) \quad (5.3)$$

by using the Kummer sequence and Galois cohomology.

Lemma 5.1. *The group $\text{III}_S(\mathbf{T})$ is finite.*

Proof. Let K be a finite Galois extension of F such that $(T_F)_K$ is a split torus over K . It is easy to see that the kernel of the canonical map $\text{III}_S(\mathbf{T}) \rightarrow \text{III}_{S_K}(\mathbf{T} \times_{\mathfrak{o}_{K, S_K}} \mathfrak{o}_{K, S_K})$, where S_K is the set of primes of K above S , is contained in $H^1(K/F, \mathbf{T}(\mathfrak{o}_{K, S}))$. Since $\mathbf{T}(\mathfrak{o}_{K, S}) \subseteq \mathbf{T}(\mathfrak{o}_{K, S_1})$ is a finitely generated abelian group, $H^1(K/F, \mathbf{T}(\mathfrak{o}_{K, S}))$ is finite. Therefore the finiteness of $\text{III}_{S_K}(\mathbf{T} \times_{\mathfrak{o}_S} \mathfrak{o}_{K, S})$ implies that of $\text{III}_S(\mathbf{T})$ and we can assume that T_F is a split torus over F . Enlarging S if necessary (which will increase $\text{III}_S(\mathbf{T})$), we can suppose that $S = S_1$ and $\mathbf{T} \cong \mathbb{G}_{m, \mathfrak{o}_S}^d$, and even that $d = 1$.

The short exact sequence

$$0 \rightarrow \mathbf{T}(\bar{\mathfrak{o}}_S) \rightarrow T_F(\bar{F}) \rightarrow T_F(\bar{F})/\mathbf{T}(\bar{\mathfrak{o}}_S) \rightarrow 0$$

gives the diagram of the following long exact sequence

$$\begin{array}{ccccccc} T_F(F)/\mathbf{T}(\mathfrak{o}_S) & \longrightarrow & (T_F(\bar{F})/\mathbf{T}(\bar{\mathfrak{o}}_S))^G & \longrightarrow & H^1(F, \mathbf{T}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_F(F_v)/\mathbf{T}(\mathfrak{o}_v) & \longrightarrow & (T_F(\bar{F}_v)/\mathbf{T}(\bar{\mathfrak{o}}_v))^{G_v} & \longrightarrow & H^1(F_v, \mathbf{T}) \longrightarrow 0 \end{array}$$

for any $v \notin S$ by Galois cohomology, where $G = \text{Gal}(\bar{F}/F)$ and $G_v = \text{Gal}(\bar{F}_v/F_v)$. Since for all $v \notin S$, G acts transitively on the primes of $\bar{\mathfrak{o}}_S$ dividing v , one has

$$\text{Ker}((T_F(\bar{F})/\mathbf{T}(\bar{\mathfrak{o}}_S))^G \rightarrow \prod_{v \notin S} (T_F(\bar{F}_v)/\mathbf{T}(\bar{\mathfrak{o}}_v))) = 0.$$

Note that the canonical map $(T_F(\bar{F})/\mathbf{T}(\bar{\mathfrak{o}}_S))^G \rightarrow \prod_{v \notin S} (T_F(\bar{F}_v)/\mathbf{T}(\bar{\mathfrak{o}}_v))$ takes values in the direct sum. Thus by the snake lemma, and because $\mathbf{T} = \mathbb{G}_{m, \mathfrak{o}_S}$,

$$\text{III}_S(\mathbf{T}) \hookrightarrow T_F(F) \setminus \bigoplus_{v \notin S} (T_F(F_v)/\mathbf{T}(\mathfrak{o}_v)) \cong \text{Pic}(\mathfrak{o}_S)$$

is finite. \square

Let \mathbf{T} be a commutative group scheme separated of finite type over \mathfrak{o}_{S_1} satisfying condition (5.1) as before. Consider the projective systems $(\mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}))_N$ and $(\text{Sel}_S^{(N)}(F, \mathbf{T}))_N$, where for any pair of natural integers $N \mid N'$, the transition map of the first system is the canonical quotient map, and that of the second system is given by the multiplication-by- N'/N map

$$\text{Sel}_S^{(N')}(F, \mathbf{T}) \xrightarrow{N'/N} \text{Sel}_S^{(N)}(F, \mathbf{T}).$$

Consider the limits

$$\widehat{\mathbf{T}(\mathfrak{o}_{S_1})} := \varprojlim_N \mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}) \quad \text{and} \quad \widehat{\text{Sel}_S(F, \mathbf{T})} := \varprojlim_N \text{Sel}_S^{(N)}(F, \mathbf{T}).$$

Corollary 5.2. *With the above notations, we have a canonical isomorphism*

$$\widehat{\mathbf{T}(\mathfrak{o}_{S_1})} \cong \widehat{\text{Sel}_S(F, \mathbf{T})}.$$

Proof. This follows from the short exact sequence (5.2) and Lemma 5.1. \square

The next lemma is proved by similar arguments to those of Lemmas 1.3-1.5 in [26].

Lemma 5.3. *There is a positive integer h depending only on \mathbf{T} such that*

$$\mathbf{T}(\mathfrak{o}_{S_1}) \cap \left(hN \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) \right) \subseteq N\mathbf{T}(\mathfrak{o}_{S_1})$$

for any positive integer N .

Proof. Let K be a finite Galois extension of F such that $T_F = \mathbf{T} \times_{\mathfrak{o}_{S_1}} F$ splits over K , let S' be the places of K above S_1 and let $\mathbf{T}' = \mathbf{T} \times_{\mathfrak{o}_{S_1}} \mathfrak{o}_{S'}$. Let G_K be the absolute Galois group of K . Then $\mathbf{T}'[N](\bar{\mathfrak{o}}_{S'})$ is a sub- G_K -module of $T_F[N](\bar{K}) = \mu_{N,K}^d(\bar{K})$. It follows that $\mathbf{T}'[N](\bar{\mathfrak{o}}_{S'})$ is isomorphic to a product of $\mu_{m,K}$'s. We have the following commutative diagram

$$\begin{array}{ccccc} H^1(K/F, \mathbf{T}[N](\mathfrak{o}_{S'})) & \longrightarrow & H^1(F, \mathbf{T}[N]) & \longrightarrow & H^1(K, \mathbf{T}'[N]) \\ & & \downarrow & & \downarrow \\ & & \prod_{v \notin S_1} H^1(F_v, \mathbf{T}[N]) & \longrightarrow & \prod_{w|v, v \notin S_1} H^1(K_w, \mathbf{T}'[N]). \end{array}$$

As $H^1(K/F, \mathbf{T}[N](\mathfrak{o}_{S'}))$ is killed by $\gcd([K : F], r_K)$, where r_K is the number of roots of unity inside K , and the kernel of the last vertical arrow is killed by 2 ([17], Theorem (9.1.9)(ii)), the kernel

$$\mathrm{Ker} \left(H^1(F, \mathbf{T}[N]) \rightarrow \prod_{v \notin S_1} H^1(F_v, \mathbf{T}[N]) \right)$$

is killed by $c := 2 \gcd([K : F], r_K)$ for any positive integer N . Let $h = 2cr_K$ which depends only on \mathbf{T} .

For any $x \in \mathbf{T}(\mathfrak{o}_{S_1}) \cap (hN \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v))$, the following commutative diagram of Kummer exact sequences

$$\begin{array}{ccccc} \mathbf{T}(\mathfrak{o}_{S_1}) & \xrightarrow{\cdot hN} & \mathbf{T}(\mathfrak{o}_{S_1}) & \longrightarrow & H^1(F, \mathbf{T}[hN]) \\ \downarrow & & \downarrow & & \downarrow \\ \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) & \xrightarrow{\cdot hN} & \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) & \longrightarrow & \prod_{v \notin S_1} H^1(F_v, \mathbf{T}[hN]) \end{array}$$

implies that there exists $y \in \mathbf{T}(\mathfrak{o}_{S_1})$ such that $cx = hNy$. Let

$$z = x - 2r_K Ny \in \mathbf{T}(\mathfrak{o}_{S_1}).$$

We have

$$z \in 2r_K N \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) \subseteq 2r_K N \prod_{w \notin S'} T_K(K_w)$$

where $T_K := T_F \times_F K$. By [17], Theorem (9.1.11)(i), there exists $\xi \in T_K(K) = (K^\times)^d$ such that $z = r_K N \xi$. As z is a c -torsion point, ξ is also a torsion point. Since r_K kills all roots of unity in K , z is trivial and $x = 2r_K Ny \in N\mathbf{T}(\mathfrak{o}_{S_1})$. The proof is complete. \square

Proposition 5.4. *Let $\overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ be the topological closure of $\mathbf{T}(\mathfrak{o}_{S_1})$ inside $T_F(\mathbb{A}_F^S)$. Then the natural homomorphisms*

$$\mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}) \rightarrow \overline{\mathbf{T}(\mathfrak{o}_{S_1})}/(\overline{\mathbf{T}(\mathfrak{o}_{S_1})}) \cap N \left(\prod_{v \in S_1 \setminus S} T_F(F_v) \times \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) \right)$$

induce an isomorphism of topological groups $\widehat{\mathbf{T}(\mathfrak{o}_{S_1})} \cong \overline{\mathbf{T}(\mathfrak{o}_{S_1})}$. In particular, the inclusion $\mathbf{T}(\mathfrak{o}_{S_1}) \subset \overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ induces an equality of the torsion parts

$$\mathbf{T}(\mathfrak{o}_{S_1})_{\mathrm{tors}} = (\overline{\mathbf{T}(\mathfrak{o}_{S_1})})_{\mathrm{tors}}.$$

Proof. Let

$$A_N = (\mathbf{T}(\mathfrak{o}_{S_1}) \cap N \left(\prod_{v \in S_1 \setminus S} T_F(F_v) \times \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) \right)) / N\mathbf{T}(\mathfrak{o}_{S_1})$$

for any positive integer N . Since $\mathbf{T}(\mathfrak{o}_{S_1})$ is a finitely generated abelian group, the quotient $\mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1})$ is finite for all N . As the canonical map

$$\mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}) \rightarrow \overline{\mathbf{T}(\mathfrak{o}_{S_1})}/(\overline{\mathbf{T}(\mathfrak{o}_{S_1})}) \cap N \left(\prod_{v \in S_1 \setminus S} T_F(F_v) \times \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) \right)$$

has dense image, it is surjective. Therefore one has the following exact sequence

$$0 \rightarrow A_N \rightarrow \mathbf{T}(\mathfrak{o}_{S_1})/N\mathbf{T}(\mathfrak{o}_{S_1}) \rightarrow \overline{\mathbf{T}(\mathfrak{o}_{S_1})}/(\overline{\mathbf{T}(\mathfrak{o}_{S_1})}) \cap N \left(\prod_{v \in S_1 \setminus S} T_F(F_v) \times \prod_{v \notin S_1} \mathbf{T}(\mathfrak{o}_v) \right) \rightarrow 0$$

for any positive integer N . Moreover, the inverse system $(A_N)_N$ satisfies the Mittag-Leffler condition by the finiteness of A_N . By taking the inverse limits, one obtains the exact sequence

$$0 \rightarrow \varprojlim_N A_N \rightarrow \widehat{\mathbf{T}(\mathfrak{o}_{S_1})} \rightarrow \overline{\widehat{\mathbf{T}(\mathfrak{o}_{S_1})}} = \overline{\mathbf{T}(\mathfrak{o}_{S_1})} \rightarrow 0,$$

the last equality follows from [18], Theorem 1.1. On the other hand, $\varprojlim_N A_N = 0$ by Lemma 5.3 and the desired isomorphism is proved. The equality on the torsion parts then follows easily from the fact that $\mathbf{T}(\mathfrak{o}_{S_1})$ is a finitely generated abelian group. \square

Recall that in the Galois cohomology groups, $\mathbf{T}[N]$ stands for the Galois module $\mathbf{T}[N](\bar{\mathfrak{o}}_{S_1})$.

Lemma 5.5. *There exists a positive integer c depending only on \mathbf{T} such that c kills all $H^1(F_N/F, \mathbf{T}[N])$ for all positive integers N , where $F_N = F(\mathbf{T}[N])$ is the smallest extension of F such that $\mathbf{T}[N](\bar{\mathfrak{o}}_{S_1}) \subseteq T_F(F_N)$.*

Proof. Let K be a finite Galois extension of F containing F_N . Then $\text{Gal}(K/F_N)$ acts trivially on $\mathbf{T}[N](\bar{\mathfrak{o}}_{S_1})$ and we have an exact sequence

$$0 \rightarrow H^1(F_N/F, \mathbf{T}[N]) \rightarrow H^1(K/F, \mathbf{T}[N]) \rightarrow H^1(K/F_N, \mathbf{T}[N]).$$

Taking K for the compositum of F_N with the splitting field of the torus T_F , we are reduced to the case when T_F is split.

Define the Galois module ν_N by the exact sequence

$$0 \rightarrow \mathbf{T}[N](\bar{\mathfrak{o}}_S) \rightarrow T_F[N](\bar{F}) = \mu_{N,F}^d \rightarrow \nu_N \rightarrow 0.$$

Then one has

$$\nu_N(F) \rightarrow H^1(F(\mu_N)/F, \mathbf{T}[N]) \rightarrow H^1(F(\mu_N)/F, \mu_N^d)$$

and

$$0 \rightarrow H^1(F_N/F, \mathbf{T}[N]) \rightarrow H^1(F(\mu_N)/F, \mathbf{T}[N])$$

by Galois cohomology. Let r_F be the number of roots of unity inside F . Then $\nu_N(F)$ is killed by r_F and one only needs to prove that $H^1(F(\mu_N)/F, \mu_N)$ is killed by a positive integer which is independent of N .

Let $N = \prod_{i=1}^t p_i^{m_i}$ be the prime factorization of N . Then one has

$$H^1(F(\mu_N), \mu_N) = \prod_{i=1}^t H^1(F(\mu_N)/F, \mu_{p_i^{m_i}})$$

and the exact sequence

$$0 \rightarrow H^1(F(\mu_{p_i^{m_i}})/F, \mu_{p_i^{m_i}}) \rightarrow H^1(F(\mu_N)/F, \mu_{p_i^{m_i}}) \rightarrow H^1(F(\mu_N)/F(\mu_{p_i^{m_i}}), \mu_{p_i^{m_i}})^{G_i}$$

where $G_i = \text{Gal}(F(\mu_{p_i^{m_i}})/F)$. Since

$$H^1(F(\mu_N)/F(\mu_{p_i^{m_i}}), \mu_{p_i^{m_i}})^{G_i} = \text{Hom}(\text{Gal}(F(\mu_N)/F(\mu_{p_i^{m_i}})), \mu_{p_i^{m_i}}(F)),$$

one concludes that $H^1(F(\mu_N), \mu_N)$ is killed by $2r_F$ by [17], Proposition 9.1.6. \square

The following lemma is an application of the Chebotarev density theorem. Recall that Equation (5.3) gives a canonical map

$$\text{Sel}_S^{(N)}(F, \mathbf{T}) \rightarrow \mathbf{T}(\mathfrak{o}_v)/N\mathbf{T}(\mathfrak{o}_v)$$

for all $v \notin S_1$.

Lemma 5.6. *Let c be a positive integer as in Lemma 5.5. Let $Q \in \text{Sel}_S^{(N)}(F, \mathbf{T})$ and let n be the order of cQ inside $\text{Sel}_S^{(N)}(F, \mathbf{T})$. Then the density of the following set*

$\{v \notin S_1 \mid v \text{ splits completely in } F_N/F \text{ and the image of } Q \text{ in } \mathbf{T}(\mathfrak{o}_v)/N\mathbf{T}(\mathfrak{o}_v) \text{ is } 0\}$
inside Ω_F is less than or equal to $1/(n \cdot [F_N : F])$, where $F_N = F(\mathbf{T}[N])$.

Proof. As $\text{Sel}_S^{(N)}(F, \mathbf{T}) \subseteq \text{Sel}_{S_1}^{(N)}(F, \mathbf{T})$, one can replace S with S_1 and suppose that $S = S_1$. Let S' be the primes of F_N above the primes of S and let

$$\text{Sel}_S^{(N)}(F_N, \mathbf{T}) = \text{Ker}(H^1(F_N, \mathbf{T}) \rightarrow \prod_{w \notin S'} H^1(F_{N,w}, \mathbf{T})).$$

The restriction map of Galois cohomology induces a homomorphism

$$\text{Sel}_S^{(N)}(F, \mathbf{T}) \xrightarrow{\text{res}} \text{Sel}_S^{(N)}(F_N, \mathbf{T})$$

whose kernel is killed by c by construction (Lemma 5.5). This implies that the order of the image of Q in $\text{Sel}_S^{(N)}(F_N, \mathbf{T})$ is a multiple of n . Since

$$\text{Sel}_S^{(N)}(F_N, \mathbf{T}) \subseteq H^1(F_N, \mathbf{T}[N]) = \text{Hom}_{\text{cont}}(\text{Gal}(\bar{F}/F_N), \mathbf{T}[N]),$$

Q gives rise to a homomorphism $\alpha \in \text{Hom}_{\text{cont}}(\text{Gal}(\bar{F}/F_N), \mathbf{T}[N])$ of order a multiple of n . Let $L = \bar{F}^{\text{Ker}(\alpha)}$. This is a Galois extension of F_N of degree a multiple of n .

If $v \notin S$ splits completely in F_N/F and w is a prime in F_N above v , then one has the following commutative diagram with injective row maps

$$\begin{array}{ccc} \text{Sel}_S^{(N)}(F_N, \mathbf{T}) & \longrightarrow & \text{Hom}_{\text{cont}}(\text{Gal}(\bar{F}/F_N), \mathbf{T}[N]) \\ \downarrow & & \downarrow \\ \mathbf{T}(\mathfrak{o}_v)/N\mathbf{T}(\mathfrak{o}_v) \cong \mathbf{T}(\mathfrak{o}_w)/N\mathbf{T}(\mathfrak{o}_w) & \longrightarrow & \text{Hom}_{\text{cont}}(\text{Gal}(\overline{F_{N,w}}/F_{N,w}), \mathbf{T}[N]) \end{array}$$

where $\overline{F_{N,w}}$ is the algebraic closure of $F_{N,w}$. So the image of Q in $\mathbf{T}(\mathfrak{o}_v)/N\mathbf{T}(\mathfrak{o}_v)$ is trivial if and only if the decomposition group at any prime of L above w is trivial. The latter is equivalent to w being split in L/F_N completely. Therefore the set of primes of F we consider is the same as the set of primes in F , not in S , and which split completely in L . By the Chebotarev density theorem, the density of this set is at most

$$\frac{1}{[L : F]} = \frac{1}{[L : F_N] \cdot [F_N : F]} \leq \frac{1}{n \cdot [F_N : F]}$$

and the proof is complete. \square

The main result of this section is the following theorem.

Theorem 5.7. *If Z is a finite closed subset of a torus T_F over F , then*

$$\overline{T_F(F)}^S \cap Z(\mathbb{A}_F^S) = Z(F)$$

where $\overline{T_F(F)}^S$ is the topological closure of $T_F(F)$ inside $T_F(\mathbb{A}_F^S)$.

Proof. We first reduce to the case $Z(F) = Z(\bar{F})$ using a similar trick to the proof of [19], Proposition 3.9. Consider a finite extension K/F such that $Z(K) = Z(\bar{F})$. Let S' be the primes of K above S . If one can show that

$$\overline{T_F(K)}^{S'} \cap Z(\mathbb{A}_K^{S'}) = Z(K),$$

then

$$Z(F) \subseteq \overline{T_F(F)}^S \cap Z(\mathbb{A}_F^S) \subseteq (\overline{T_F(K)}^{S'} \cap Z(\mathbb{A}_K^{S'})) \cap Z(\mathbb{A}_F^S) = Z(K) \cap Z(\mathbb{A}_F^S) = Z(F),$$

the last equality being proved similarly to Lemma 3.2 in [19], and the desired equality will hold over F .

Since $Z(F)$ is finite, there exists a finite subset $S_1 \supset S$ such that T extend to an affine group scheme \mathbf{T} of finite type over \mathfrak{o}_{S_1} and $Z(F) \subset \mathbf{T}(\mathfrak{o}_{S_1})$ and the condition (5.1) is satisfied. One only needs to show that

$$\overline{\mathbf{T}(\mathfrak{o}_{S_1})} \cap Z(\mathbb{A}_F^S) \subseteq Z(F)$$

where $\overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ is the topological closure of $\mathbf{T}(\mathfrak{o}_{S_1})$ inside $T_F(\mathbb{A}_F^S)$. Indeed, the left-hand side contains $\overline{T_F(F)}^S \cap Z(\mathbb{A}_F^S)$ as we saw in the proof of Corollary 4.7.

For any

$$(x_v)_{v \notin S} \in \overline{\mathbf{T}(\mathfrak{o}_{S_1})} \cap Z(\mathbb{A}_F^S),$$

we claim that there is $z \in Z(F)$ such that $(x_v)_{v \notin S} - z$ is torsion. Suppose the contrary. Then all $(x_v)_{v \notin S} - z$ with $z \in Z(F)$ are of infinite order. Fix a positive integer n with $n > \#Z(F)$. Use Proposition 5.4 and Corollary 5.2 to identify $\overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ to a subgroup of $\text{Sel}_S(F, \mathbf{T})$. By the same argument as [23], Lemma 3.5, there is a positive integer N such that the order of $c((x_v)_{v \notin S} - z)$ in $\text{Sel}_S^{(N)}(F, \mathbf{T})$ is at least n for all $z \in Z(F)$, here c is the positive integer defined Lemma 5.5. By Lemma 5.6, the density of primes $v \in \Omega_F \setminus S_1$ such that v splits completely in F_N/F and at least one of $(x_v)_{v \notin S} - z$ is trivial in $\mathbf{T}(\mathfrak{o}_v)/N\mathbf{T}(\mathfrak{o}_v)$ for some $z \in Z(F)$ is at most

$$\frac{\#Z(F)}{n[F_N : F]} < \frac{1}{[F_N : F]}.$$

This implies that there is $v_0 \notin S_1$ such that v_0 splits completely in F_N/F and none of $x_{v_0} - z$ is trivial in $\mathbf{T}(\mathfrak{o}_{v_0})/N\mathbf{T}(\mathfrak{o}_{v_0})$ for all $z \in Z(F)$. In particular, one has $x_{v_0} \neq z$ for all $z \in Z(F) = Z(F_{v_0})$. This contradicts the hypothesis $(x_v)_{v \notin S} \in Z(\mathbb{A}_F^S)$.

Since the torsion subgroup of $\overline{\mathbf{T}(\mathfrak{o}_{S_1})}$ is the same as the torsion subgroup $\mathbf{T}(\mathfrak{o}_{S_1})_{\text{tor}}$ of $\mathbf{T}(\mathfrak{o}_{S_1})$ by Proposition 5.4, one concludes that

$$(x_v)_{v \notin S} \in Z(F) + \mathbf{T}(\mathfrak{o}_{S_1})_{\text{tor}} \subseteq \mathbf{T}(\mathfrak{o}_{S_1}) \subset T_F(F).$$

The result follows from $Z(\mathbb{A}_F^S) \cap T_F(F) = Z(F)$. \square

Next we make a “ S -adelic Mordell-Lang conjecture” for tori in the same spirit of Question 3.12 in [23] which was proved for abelian varieties over a global function field in [19].

Let X_F be a hyperbolic affine curve contained in \mathbf{P}_F^1 and let $j : X_F \rightarrow T_F$ be a closed immersion as in Lemma 4.4 with a torus T_F .

Conjecture 5.8. *If G is a finitely generated subgroup of $T_F(F)$ and \overline{G}^S is the topological closure of G inside $T_F(\mathbb{A}_F^S)$, then there is a finite subscheme $Z \subset X$ such that*

$$X(\mathbb{A}_F^S) \cap \overline{G}^S \subseteq Z(\mathbb{A}_F^S).$$

The Harari-Voloch conjecture over number fields will follow from this conjecture and Theorem 5.7 (see Question 3.12 in [23]).

As an application of Theorem 5.7, we show that Theorem 3 in [12] holds for any quasi-finite morphisms of rational hyperbolic curves (see Remarks 4.2 and 5.10).

Proposition 5.9. *Suppose $f : X_F \rightarrow Y_F$ is a quasi-finite morphism of curves over F and X_F is a rational curve inside \mathbf{P}_F^1 with complement of degree ≥ 2 . Fix a finite subset S of primes of F containing ∞_F . If $Y_F(\mathbb{A}_F^S)^{B_S(Y)} = Y_F(F)$, then $X_F(\mathbb{A}_F^S)^{B_S(X)} = X_F(F)$.*

Proof. For any $x \in X_F(\mathbb{A}_F^S)^{B_S(X)}$, there is $y \in Y_F(F)$ such that $f(x) = y$ by the functoriality of the Brauer-Manin pairing and the assumption. Then $Z = f^{-1}(y)$ is a finite closed subscheme over F inside X_F and $x \in Z(\mathbb{A}_F^S)$. Let $j : X_F \rightarrow T_F$ be the closed immersion given by Lemma 4.4. Let $\overline{T_F(F)}^S$ be the closure of $T_F(F)$ in $T_F(\mathbb{A}_F^S)$. By Lemma 4.3 (2), see also Definition 4.5, the immersion $X_F \rightarrow T_F$ induces a map

$$X_F(\mathbb{A}_F^S)^{B_S(X)} \longrightarrow T_F(\mathbb{A}_F^S)^{B_{1,S}(T_F)}.$$

By Proposition 4.6 (1), one has

$$(T_F(\mathbb{A}_F^S))^{B_{1,S}(T_F)} = \overline{T_F(F)}^S,$$

hence

$$x \in \overline{T_F(F)}^S \cap Z(\mathbb{A}_F^S) = Z(F) \subseteq X_F(F)$$

by Theorem 5.7. The proof is complete. \square

Remark 5.10 Let C_F be a projective smooth curve of positive genus over F . In [23], Theorem 8.2, it is proved that for any finite closed subset Z of C_F , one has

$$Z(\mathbb{A}_F) \cap C_F(\mathbb{A}_F)_{\bullet}^{\text{Br}(C_F)} = Z(F),$$

where the subscript \bullet means that at infinite places v , $C_F(F_v)$ is replaced with its set of connected components $\pi_0(C_F(F_v))$.

Using the above result, J.-L. Colliot-Thélène proved in [2] that [12] Theorem 3, can be generalized to covers of arbitrary degree, provided the group $B_S(\cdot)$ is replaced by the whole Brauer group $\text{Br}(\cdot)$. Our proof of Proposition 5.9 is based on similar arguments. If X_F is an open subvariety of C_F containing Z and if the analogue equality

$$Z(\mathbb{A}_F^S) \cap X_F(\mathbb{A}_F)^{B_S(X_F)} = Z(F)$$

holds, then one can remove the rationality hypothesis on X_F in Proposition 5.9.

Corollary 5.11. *Let D be a reduced effective divisor in $\mathbf{P}_{\mathbb{Q}}^1$ supported in at least two points. Let $X_{\mathbb{Q}}$ be the complement of D and let $S = \{\infty_{\mathbb{Q}}\}$. Then*

$$X_{\mathbb{Q}}(\mathbb{Q}) = X_{\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}}^S)^{B_S(X_{\mathbb{Q}})}.$$

In particular, Harari-Voloch conjecture holds for $X_{\mathbb{Q}}$ and S if $\deg D \geq 3$ (equivalently, if $X_{\mathbb{Q}}$ is hyperbolic).

Proof. Let P_1, P_2 be two points in D . There exists a non-constant invertible element $f \in \mathcal{O}(X_{\mathbb{Q}})^{\times}$. The finite morphism $\mathbf{P}_{\mathbb{Q}}^1 \rightarrow \mathbf{P}_{\mathbb{Q}}^1$ defined by f induces a quasi-finite morphism $X_{\mathbb{Q}} \rightarrow Y_{\mathbb{Q}} := \mathbb{G}_{m, \mathbb{Q}}$. As

$$Y_{\mathbb{Q}}(\mathbb{A}_{\mathbb{Q}}^S)^{B_S(Y_{\mathbb{Q}})} = Y_{\mathbb{Q}}(\mathbb{Q})$$

(e.g. by Proposition 4.6 (2) and Theorem 3.5 (2)), the same equality holds for $X_{\mathbb{Q}}$ by Proposition 5.9. \square

Acknowledgements. We would like to thank J.-L. Colliot-Thélène, D. Harari, Y. Liang and F. Voloch for helpful discussions. We would also like to thank Liang-Chung Hsia for drawing our attention to [24]. We thank the referee for a careful reading of the manuscript and for pointing out some inaccuracies. The first named author thanks Capital Normal University, where part of this work was done, for its support. The second named author is supported by the ALGANT program in Université de Bordeaux, MPI for mathematics at Bonn from Sep.-Oct.2014 and NSFC grant no. 11031004.

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